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On finite amplitude oscillations in laminar mixing layers

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In the first part of the paper, a mixing layer of tanh y form is considered, and two-dimensional solutions of the non-linear inviscid equations are found representing periodic perturbations from the neutral wave of linearized stability theory. To second order in amplitude the solutions are equivalent to the equilibrium state calculated by Schade (1964), who studied the development of perturbations in time and found an evolution towards that equilibrium state. The present calculation, however, is taken to fourth-order in amplitude. It is noted that the solutions presented in this paper are regular, even though viscosity is ignored; and the relationships to the singular (if inviscid) time-dependent solutions of Schade are explained. Such regular, inviscid solutions have been found only for odd velocity profiles, such as tanh y.

Although the details of the velocity distributions are not of the form observed experimentally, it is shown that the amplitude ratios of fundamental and first harmonic, for a given absolute amplitude, are comparable with those observed.

In part 2 some exact non-linear solutions are presented of the inviscid, incompressible equations of fluid flow in two or three spatial dimensions. They illustrate the flows of part 1, since they are periodic in one co-ordinate (x), have a shear in another (y) and are independent of the third. Included, as two-dimensional cases, are (i) the tanh y velocity distribution for a flow wholly in the x-direction, (ii) the well-known solution for the flow due to a set of point vortices equi-spaced on the axis, and (iii) an example of linearized hydrodynamic (Orr–Sommerfeld) stability theory. The flows may involve concentrations of vorticity. In three-dimensional cases the z component of velocity is even in y, whereas the x component is odd. Consequently, the class of flows represents, in general, small or large periodic perturbations from a skewed shear layer. Time-dependent solutions, representing waves travelling in the x direction may be obtained by translation of axes.

1. Introduction

Schade (1964) has considered the application of current ideas in non-linear hydrodynamic stability theory to the flow in a tanh y shear layer. He started from an unstable perturbation given by linearized stability theory, the wave-number chosen (α) being less than the value (unity) appropriate to a neutral disturbance (neither amplifying nor decaying) at infinite Reynolds number (see figure 1, where A is the neutral wave-number and B the chosen wave-number, α). By incorporating non-linear effects he then derived an equation for the growth with time of the perturbation, and calculated an amplitude at which the perturbation equilibrated in a natural manner; in the calculation it was assumed that α was close to the neutral value of unity, this being a way of ensuring small amplitudes. A difficulty which arose in the analysis was connected with the fact that the linearized-theory equations for non-neutral perturbations are singular; this led Schade to introduce viscosity, as described by Lin (1955), to smooth out the effect of that singularity on the perturbation velocity distributions.

It is emphasized here that another perturbation scheme, not involving singularities and not requiring the introduction of viscosity, can be used to calculate the equilibrium amplitude and the flow field for small, but non-zero, perturbations from the $\tanh y$ profile. Instead of following the development of perturbations with time, as Schade did, we may expand both the velocity and the wave-number in powers of a small parameter ϵ, and thereby obtain finite-amplitude steady solutions at wave-numbers less than the neutral value. As emphasized elsewhere (Stuart 1960) the two approaches should, subject to related assumptions being made, yield the same solution. This is shown to be the case in the present paper, and in making the comparison with Schade's work we are able to clarify certain aspects of his theory. Especially it becomes clear that the effect of viscosity, which is vital in his solution at small amplitude, dies out at larger amplitudes as the flow approaches equilibrium, at least in the sense that viscosity ceases to appear explicitly in the solution.

Both Schade's solution and that of the present perturbation scheme are calculated for the case in which the mean flow (averaged in the flow direction) is left unperturbed. Other possibilities could be investigated by similar means, and which case really occurred would depend on matters such as initial conditions and boundary-layer growth, both these factors being ignored in Schade's work and the present paper. It happens, however, that there are some exact solutions of the nonlinear inviscid flow equations, related to those described in this paper but involving mean-velocity changes, and these are described in part 2.

The relevance to experiment of the solutions described is considered in the last section of the present paper.

Part 1. Perturbation Theory

2. Equations of motion and a method of solution

We consider an infinite region of fluid, which is inviscid and incompressible. Velocity components, lengths and time are made non-dimensional in some convenient way. Then the vorticity equation for two-dimensional motion of the inviscid incompressible fluid is of the form

$$\frac{\partial}{\partial t}\nabla^2\psi+\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi = 0, \tag{2.1}$$

where ψ is the stream function, t is the time, x is the co-ordinate in the direction of mean flow and y is the co-ordinate normal to that direction; x and y velocity components are $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. We shall consider solutions that are periodic in x and time, and for this reason it is desirable to introduce a wave-number (α) and a wave propagation speed (c). This we do by defining

$$\alpha x = \xi, \quad \alpha c t = \tau, \tag{2.2}$$

and then equation (2.1) takes the form

$$c\frac{\partial}{\partial \tau}(\psi'' + \alpha^2\psi_{\xi\xi}) + \psi(\psi''_\xi + \alpha^2\psi_{\xi\xi\xi}) - \psi_\xi(\psi''' + \alpha^2\psi'_{\xi\xi}) = 0, \tag{2.3}$$

where we have divided by α. A prime denotes differentiation with respect to y, and a suffix ξ differentiation with respect to ξ.

Let us now look for a solution of (2.3) in the form of a perturbation series:

$$\psi = \psi_0(y) + \epsilon\psi_1(y,\xi) + \epsilon^2\psi_2(y,\xi) + \epsilon^3\psi_3(y,\xi) + \ldots, \tag{2.4}$$

$$\alpha^2 = \alpha_0^2 + \epsilon\alpha_1 + \epsilon^2\alpha_2 + \epsilon^3\alpha_3 + \epsilon^4\alpha_4 + \ldots, \tag{2.5}$$

$$c = c_0 + \epsilon c_1 + \epsilon^2 c_2 + \ldots. \tag{2.6}$$

Substituting (2.4), (2.5) and (2.6) in (2.3), and separating out powers of ϵ, we obtain

$$O(\epsilon)\colon \left(c_0\frac{\partial}{\partial\tau} + \psi'_0\frac{\partial}{\partial\xi}\right)(\psi''_1 + \alpha_0^2\psi_{1\xi\xi}) - \psi'''_0\,\psi_{1\xi} = 0, \tag{2.7}$$

$$\begin{aligned} O(\epsilon^2)\colon \left(c_0\frac{\partial}{\partial\tau} + \psi'_0\frac{\partial}{\partial\xi}\right)&(\psi''_2 + \alpha_0^2\psi_{2\xi\xi} + \alpha_1\psi_{1\xi\xi}) - \psi'''_0\,\psi_{2\xi} \\ &+ c_1\frac{\partial}{\partial\tau}(\psi''_1 + \alpha_0^2\psi_{1\xi\xi}) + \left(\psi'_1\frac{\partial}{\partial\xi} - \psi_{1\xi}\frac{\partial}{\partial y}\right)(\psi''_1 + \alpha_0^2\psi_{1\xi\xi}) = 0, \end{aligned} \tag{2.8}$$

$$\begin{aligned} O(\epsilon^3)\colon \left(c_0\frac{\partial}{\partial\tau} + \psi'_0\frac{\partial}{\partial\xi}\right)&(\psi''_3 + \alpha_0^2\psi_{3\xi\xi} + \alpha_1\psi_{2\xi\xi} + \alpha_2\psi_{1\xi\xi}) - \psi'''_0\,\psi_{3\xi} \\ &+ c_1\frac{\partial}{\partial\tau}(\psi''_2 + \alpha_0^2\psi_{2\xi\xi}) + c_2\frac{\partial}{\partial\tau}(\psi''_1 + \alpha_0^2\psi_{1\xi\xi}) \\ &+ \left(\psi'_1\frac{\partial}{\partial\xi} - \psi_{1\xi}\frac{\partial}{\partial y}\right)(\psi''_2 + \alpha_0^2\psi_{2\xi\xi} + \alpha_1\psi_{1\xi\xi}) \\ &+ \left(\psi'_2\frac{\partial}{\partial\xi} - \psi_{2\xi}\frac{\partial}{\partial y}\right)(\psi''_1 + \alpha_0^2\psi_{1\xi\xi}) = 0, \end{aligned} \tag{2.9}$$

with related equations for higher-order amplitudes. (We shall consider the higher-order equations later for a special case.) As boundary conditions we have

$$\psi_n \to 0 \quad \text{as} \quad y \to \pm\infty \quad (n = 0, 1, 2, \ldots) \tag{2.10}$$

with the ψ_n periodic in ξ and τ.

Our object is to solve the set of equations in succession, and simultaneously to determine the coefficients α_0, α_1, α_2, c_0, c_1, c_2, etc. Let us first examine (2.7): since we are seeking for wave solutions, which are neither amplifying nor decaying, we write

$$\psi_1 = \phi_1(y)\cos(\xi - \tau). \tag{2.11}$$

(This is equivalent to $\phi_1(y)\cos\alpha(x-ct)$.) Substituting (2.11) in (2.7) we obtain

$$(\psi'_0 - c_0)(\phi''_1 - \alpha_0^2\phi_1) - \psi'''_0\,\phi_1 = 0. \tag{2.12}$$

The boundary conditions are

$$\phi_1 \to 0 \quad \text{as} \quad y \to \pm\infty. \tag{2.13}$$

Since $\psi_0' = \bar{u}_0$ is the original mean motion ($\epsilon \to 0$), (2.12) is clearly the inviscid Orr–Sommerfeld (or Rayleigh) equation. Suppose the profile $\bar{u}_0$ has a point of inflexion, at which $\psi_0''' = 0$; then, for many velocity profiles, a neutral solution of (2.11) exists with c_0 given by the value of $\bar{u}_0$ at the point of inflexion. The value of α_0^2 is chosen so that the solution satisfies the boundary conditions (2.13). One well-known solution, that for a mixing layer, is

$$\psi_0' = \tanh y, \quad \phi_1 = \operatorname{sech} y, \quad \alpha_0^2 = 1, \quad c_0 = 0, \tag{2.14}$$

while another, that for a jet, is

$$\psi_0' = \operatorname{sech}^2 y, \quad \phi_1 = \operatorname{sech}^2 y, \quad \alpha_0^2 = 4, \quad c_0 = \tfrac{2}{3}. \tag{2.15}$$

We now consider equation (2.8), and seek a solution of the form

$$\psi_2 = \phi_{21}\cos(\xi-\tau) + \phi_{22}\cos 2(\xi-\tau), \tag{2.16}$$

and we find

$$(\psi_0'-c_0)(\phi_{21}''-\alpha_0^2\phi_{21}) - \psi_0'''\phi_{21} = \alpha_1(\psi_0'-c_0)\phi_1 + c_1(\phi_1''-\alpha_0^2\phi_1), \tag{2.17}$$

$$(\psi_0'-c_0)(\phi_{22}''-4\alpha_0^2\phi_{22}) - \psi_0'''\phi_{22} = \tfrac{1}{4}(\phi_1\phi_1'''-\phi_1'\phi_1''). \tag{2.18}$$

Let us require, if possible, that our solution be regular. Since $(\psi_0'-c_0)$ has a zero the solutions of equations (2.17) and (2.18) will not be regular unless the right-hand sides have zeros to cancel that of $(\psi_0'-c_0)$. With use of (2.12) equation (2.17) can be re-written

$$\phi_{21}''-\alpha_0^2\phi_{21}+K(y)\phi_{21} = \alpha_1\phi_1 - \frac{c_1 K(y)\phi_1}{\psi_0'-c_0}, \tag{2.19}$$

where

$$K(y) = -\psi_0'''/(\psi_0'-c_0), \tag{2.20}$$

which is a regular function of y. If neither ϕ_1 nor $K(y)$ has a zero at the critical point, where $\psi_0' = c_0$, then the right-hand side is singular. We therefore set $c_1 = 0$ to avoid this possibility. If now we multiply (2.19) by the adjoint function, which is ϕ_1 because the left-hand side of (2.19) is self-adjoint, corresponding to the self-adjoint form of (2.12), integrate between the limits and apply the boundary conditions

$$\phi_{21} \to 0 \quad \text{as} \quad y \to \pm\infty, \tag{2.21}$$

we find $\alpha_1 = 0$. Thus $\phi_{21} = \text{const.}\,\phi_1$. This, however, merely repeats the eigen-solution ϕ_1 and may be omitted; if kept it would imply a re-definition of the amplitude ϵ.

From (2.12) and (2.20), (2.18) becomes

$$(\psi_0'-c_0)(\phi_{22}''-4\alpha_0^2\phi_{22}) - \psi_0'''\phi_{22} = -\tfrac{1}{4}K'(y)\phi_1^2. \tag{2.22}$$

In general the solution of this equation is singular unless the right-hand side has a zero at the critical point to counter the zero in $(\psi_0'-c_0)$. Normally ϕ_1 is not zero at this point (see, e.g. Lin 1955, p. 54), but the function $K'(y)$ is zero there for velocity profiles which are odd in a co-ordinate system chosen to move with the velocity of the inflexion point, such as that given by solution (2.14); for such odd velocity profiles, which correspond to mixing layers, we can obtain a regular solution. For jet or wake profiles, however, such as that given by solution (2.15), $K'(y)$ is not zero at the critical point, and the solution of ϕ_{22} will have a singularity. We shall concentrate in what follows on the mixing-layer profiles.

Turning now to equation (2.9) we seek a solution in the form

$$\psi_3 = \phi_{31}\cos(\xi-\tau)+\phi_{32}\cos 2(\xi-\tau)+\phi_{33}\cos 3(\xi-\tau). \tag{2.23}$$

The equations for ϕ_{31}, ϕ_{32}, and ϕ_{33} are

$$\begin{aligned}(\psi_0'-c_0)(\phi_{31}''-\alpha_0^2\phi_{31}-\alpha_2\phi_1)-\psi_0'''\phi_{31}\\ = -\phi_1'(\phi_{22}''-4\alpha_0^2\phi_{22})+\tfrac{1}{2}\phi_{22}'(\phi_1''-\alpha_0^2\phi_1)\\ +\phi_{22}(\phi_1'''-\alpha_0^2\phi_1')-\tfrac{1}{2}\phi_1(\phi_{22}'''-4\alpha_0^2\phi_{22})+c_2(\phi_1''-\alpha_0^2\phi_1),\end{aligned} \tag{2.24}$$

$$(\psi_0'-c_0)(\phi_{32}''-4\alpha_0^2\phi_{32}-4\alpha_1\phi_{22})-\psi_0'''\phi_{32}=0, \tag{2.25}$$

$$\begin{aligned}(\psi_0'-c_0)(\phi_{33}''-9\alpha_0^2\phi_{33})-\psi_0'''\phi_{33}\\ = -\tfrac{1}{3}\phi_1'(\phi_{22}''-4\alpha_0^2\phi_{22})-\tfrac{1}{6}\phi_{22}'(\phi_1''-\alpha_0^2\phi_1)\\ +\tfrac{1}{3}\phi_{22}(\phi_1'''-\alpha_0^2\phi_1')-\tfrac{1}{6}\phi_1(\phi_{22}'''-4\alpha_0^2\phi_{22}').\end{aligned} \tag{2.26}$$

We shall adopt the same criterion as previously of seeking a regular solution. If ψ_0' is odd it can be seen from (2.14) that ϕ_1 is even, since that particular property is generally true for odd velocity profiles. Moreover, (2.22) indicates that ϕ_{22} is even. Thus the right-hand sides of (2.24) and (2.26) are odd except for the term proportional to c_2 in (2.24). If c_2 is not zero, therefore, ϕ_{31} is singular, because $(\phi_1''-\alpha_0^2\phi_1)$ has no zero at the critical point to cancel that in $(\psi_0'-c_0)$. We set $c_2=0$. In order for equation (2.24) to have a solution satisfying the boundary conditions

$$\phi_{31}\to 0 \quad \text{as} \quad y\to\pm\infty, \tag{2.27}$$

it is necessary for α_2 to have a special value. This can be obtained by standard methods, with use of the adjoint function.

It will be recognized that we have sought solutions of (2.7), (2.8) and (2.9) which are periodic in ξ, which are regular, and whose mean values with respect to ξ are zero. We have excluded $O(\epsilon^n)$ $(n=1,2,3,\ldots)$ corrections to the basic mean stream function. If such corrections were included the mean stream function could be redefined (in 2.4) to include the higher-order terms; then the series would have the form described in this section, with $\psi_n(y,\xi)$ $(n=1,2,3,\ldots)$ having zero mean with respect to ξ. We shall return to this point later, especially by discussion of a class of solutions in part 2.

3. Solution for a mixing-layer profile

We now consider the solution of the equations of the last section for a special mixing-layer profile, namely

$$\bar{u}_0=\psi_0'=\tanh y; \tag{3.1}$$

for this profile the solution of equation (2.12) is given by formulae (2.14). It then follows that (2.22) takes the form

$$\phi_{22}''-4\phi_{22}+2\,\mathrm{sech}^2 y\,\phi_{22}=\mathrm{sech}^4 y, \tag{3.2}$$

subject to the boundary conditions

$$\phi_{22} \to 0 \quad \text{as} \quad y \to \pm\infty. \tag{3.3}$$

The solution is

$$\phi_{22} = -\tfrac{1}{4}\operatorname{sech}^2 y. \tag{3.4}$$

As mentioned in § 2, ϕ_{21} may be taken to be zero, so (3.4) completes the solution to second order in amplitude.

It is now possible to consider(2.24), which takes the form

$$\phi_{31}'' - \phi_{31} + 2\operatorname{sech}^2 y\,\phi_{31} = \alpha_2 \operatorname{sech} y + \tfrac{5}{2}\operatorname{sech}^5 y, \tag{3.5}$$

the boundary conditions being (2.27). The adjoint function required for the solution of (3.5) is ϕ_1, because the operator on the left-hand side of (3.5) is in self-adjoint form. If we multiply (3.5) by $\phi_1 = \operatorname{sech} y$ and integrate from $-\infty$ to $+\infty$, we find that

$$\int_{-\infty}^{\infty} \phi_1[\phi_{31}'' - \phi_{31} + 2(\operatorname{sech}^2 y)\,\phi_{31}]\,dy = \int_{-\infty}^{\infty} \phi_{31}\,[\phi_1'' - \phi_1 + 2(\operatorname{sech}^2 y)\,\phi_1]\,dy = 0, \tag{3.6}$$

since ϕ_1 satisfies (2.12). Thus, from (3.5),

$$\alpha_2 = -\frac{5}{2}\int_{-\infty}^{\infty} \operatorname{sech}^6 y\,dy \Big/ \int_{-\infty}^{\infty} \operatorname{sech}^2 y\,dy = -\frac{4}{3}. \tag{3.7}$$

Having obtained the value of α_2, we may now solve (3.5) to obtain

$$\phi_{31} = \tfrac{2}{3}\operatorname{sech} y \ln\cosh y - \tfrac{1}{4}\operatorname{sech}^3 y. \tag{3.8}$$

An arbitrary multiple of the eigenfunction may be added to (3.8), but we omit it because it may be regarded simply as altering the definition of the amplitude ϵ.

The solution for ϕ_{32} is

$$\phi_{32} = 0, \tag{3.9}$$

while the equation for ϕ_{33} is

$$\phi_{33}'' - 9\phi_{33} + 2\operatorname{sech}^2 y\,\phi_{33} = -\tfrac{5}{6}\operatorname{sech}^5 y, \tag{3.10}$$

and the boundary conditions are

$$\phi_{33} \to 0 \quad \text{as} \quad y \to \pm\infty. \tag{3.11}$$

The solution is

$$\phi_{33} = \tfrac{1}{12}\operatorname{sech}^3 y. \tag{3.12}$$

Higher-order terms in the series (2.4) can be calculated in a similar way to the solutions already described in this section, from equations like (2.9). For an odd mixing-layer profile we have deduced that regular solutions have $c = 0$ to order ϵ^2; for the calculation of the higher-order terms we now set $c \equiv 0$. Moreover, it can be argued that $\alpha_3 = 0$, just as $\alpha_1 = 0$. The higher-order solution takes the form

$$\psi_4 = \phi_{42}\cos 2\xi + \phi_{44}\cos 4\xi, \tag{3.13}$$

$$\psi_5 = \phi_{51}\cos\xi + \phi_{53}\cos 3\xi + \phi_{55}\cos 5\xi, \tag{3.14}$$

since $c \equiv 0$ implies $\tau \equiv 0$ for finite t. Detailed calculation shows that

$$\phi_{42} = -\tfrac{1}{3}\operatorname{sech}^2 y \ln\cosh y + \tfrac{1}{3}\operatorname{sech}^2 y + \tfrac{1}{4}\operatorname{sech}^4 y, \tag{3.15}$$

$$\phi_{44} = -\tfrac{1}{32}\operatorname{sech}^4 y, \tag{3.16}$$

while ϕ_{51} satisfies

$$\begin{aligned}\phi_{51}'' - \phi_{51} + 2\operatorname{sech}^2 y\phi_{51} \\ = \alpha_4 \operatorname{sech} y - \tfrac{8}{9}\operatorname{sech} y \ln\cosh y + \tfrac{1}{3}\operatorname{sech}^3 y \\ + 5\operatorname{sech}^5 y \ln\cosh y - \tfrac{2}{3}\operatorname{sech}^5 y - 14\operatorname{sech}^7 y.\end{aligned} \tag{3.17}$$

The boundary conditions on ϕ_{51} are

$$\phi_{51} \to 0 \quad \text{as} \quad y \to \pm\infty. \tag{3.18}$$

In view of the form of the left-hand side of (3.17), α_4 may be calculated just as α_2 was. We then find that

$$\alpha_4 = \tfrac{16}{3}(1 + \tfrac{1}{3}\ln 2). \tag{3.19}$$

Having calculated α_4, we know the solution for the stream function to the fourth power in amplitude. There is no point in evaluating ϕ_{51}, etc., unless we go on to calculate α_6, because the latter coefficient also contributes to the terms of fifth power in amplitude (through $\epsilon\psi_1$). The solution to $O(\epsilon^4)$ is

$$\begin{aligned}\psi = \ln\cosh y + \epsilon \operatorname{sech} y \cos\xi - \tfrac{1}{4}\epsilon^2 \operatorname{sech}^2 y \cos 2\xi \\ + \epsilon^3\{(\tfrac{2}{3}\operatorname{sech} y \ln\cosh y - \tfrac{1}{4}\operatorname{sech}^3 y)\cos\xi + \tfrac{1}{12}\operatorname{sech}^3 y \cos 3\xi\} \\ + \epsilon^4\{(-\tfrac{1}{3}\operatorname{sech}^2 y \ln\cosh y + \tfrac{1}{3}\operatorname{sech}^2 y + \tfrac{1}{4}\operatorname{sech}^4 y)\cos 2\xi \\ - \tfrac{1}{32}\operatorname{sech}^4 y \cos 4\xi\} + \ldots,\end{aligned} \tag{3.20}$$

$$\xi \equiv \alpha x, \tag{3.21}$$

$$\alpha^2 = 1 - \tfrac{4}{3}\epsilon^2 + \tfrac{16}{3}(1 + \tfrac{1}{3}\ln 2)\epsilon^4 + \ldots. \tag{3.22}$$

The latter formula can be inverted to give

$$\epsilon = \pm[\tfrac{3}{4}(1-\alpha^2)]^{\frac{1}{2}}[1 + \tfrac{1}{2}(3 + \ln 2)(1-\alpha^2) + \ldots]. \tag{3.23}$$

The coefficient α_6, if known, would yield a term of order $(1-\alpha^2)^2$ within the square bracket of (3.23), and would therefore contribute a term of order $(1-\alpha^2)^{\frac{5}{2}}$, or fifth power in amplitude, in the $O(\epsilon)$ term of (3.20).

One point that seems fairly clear is that the series (3.20) is probably not convergent for all y because, when $y \to \pm\infty$, the terms proportional to $\cos\xi$ are approximately given by

$$\epsilon \operatorname{sech} y \cos\xi . \{1 + \tfrac{2}{3}\epsilon^2(|y| - \ln 2) + \ldots\}. \tag{3.24}$$

It is seen that the term $O(\epsilon^3)$ is proportional to $|y|$ so that, for $|y|$ of order ϵ^{-2} or greater, the basic perturbation solution (3.20) must break down. Even so it is arguable that this feature affects the solution only at large values of $|y|$ $(\sim\epsilon^2)$, where (3.20) may be regarded as an invalid expansion of the true solution. In fact (3.24), which gives the dominant part of the perturbation (3.20) from $\ln\cosh y$, appears to be an expanded form, valid for $\epsilon^2|y| \ll 1$, of $\epsilon 2^\alpha e^{-\alpha|y|}\cos\xi$, which is of the form of the true perturbation solution of (2.1) as $|y| \to \infty$, in the case when any *mean* perturbation is identically zero. Other harmonics, if regarded similarly, would give further terms in an expansion valid for $|y|$ of order ϵ^2 or greater. These considerations suggest that (3.20) is a valid representation for finite y.

The series (3.23) appears to be valid in the truncated form shown only for a small range of α, say $1 \leqslant \alpha \leqslant 1-\frac{1}{40}$, corresponding to $\epsilon < \frac{1}{5}$ approximately. The range of α does not extend down to the value of α, of order 0·44 (Michalke 1965) for maximum amplification (point C of figure 1).

4. Comparison with Schade's work on non-linear growth to equilibrium

In the work so far described we have evaluated a finite-amplitude oscillatory flow in equilibrium at infinite Reynolds number. The conditions that we have applied in order to obtain this solution have included the requirements (i) that it shall be regular and (ii) that the mean part of the motion shall be unchanged. The latter condition could, as mentioned earlier, easily be relaxed, but condition (i) is more crucial.

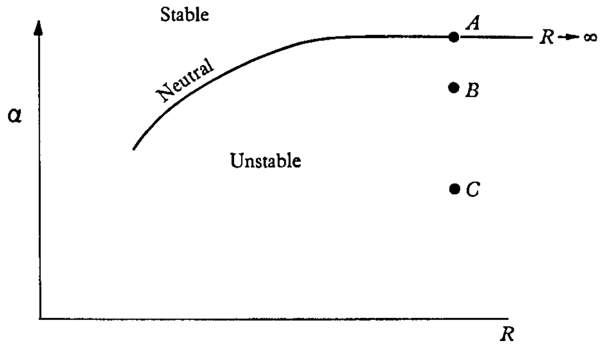

FIGURE 1. Schematic stability diagram for tanh profile.

Let us refer to figure 1 (see, for example, Betchov & Szewczyk 1963): by perturbation about the point A (on the asymptotic branch of the neutral curve with α non-zero at infinite Reynolds number), the finite-amplitude oscillation (3.20) is valid at a point B sufficiently close to A. On the other hand Schade (1964) has studied the temporal development, at a point B, of the solution which is amplified according to linearized theory. He finds that an equilibrium state of finite amplitude is attained. It is natural to suppose, as has been argued in a related problem (Stuart 1960, p. 69), that the present solution should be equivalent to Schade's solution, when the latter has reached equilibrium. Since Schade (1964) has calculated only the first (a_1) of the nonlinear coefficients in his expansion

$$dA/dt = -i\alpha c A + a_1 A\,|A|^2 + a_2 A\,|A|^4 + \ldots \tag{4.1}$$

of the amplitude $A(t)$ of the perturbation, his solution is valid only to second-order in amplitude, $O(c_i)$. This corresponds to our result $O(\epsilon^2)$, as we shall shortly see, and the comparison will be made on this basis.

To $O(\epsilon^2)$, which is equivalent to $O(1-\alpha^2)$ because of (3.22), (3.20) yields

$$\psi = \ln\cosh y + [\tfrac{3}{4}(1-\alpha^2)]^{\frac{1}{2}}\operatorname{sech} y\,\cos\alpha x - \tfrac{3}{16}(1-\alpha^2)\operatorname{sech}^2 y\,\cos 2\alpha x. \tag{4.2}$$

(We consider the positive sign only in (3.23), since the negative sign involves simply a change of phase.) By differentiation with respect to x we obtain the y-component of velocity as

$$v = \alpha[\tfrac{3}{4}(1-\alpha^2)]^{\frac{1}{2}}\operatorname{sech} y\,\sin\alpha x - \tfrac{3}{8}\alpha(1-\alpha^2)\operatorname{sech}^2 y\,\sin 2\alpha x. \tag{4.3}$$

Schade (1964) gives

$$v = A_e v_1(y)\, e^{i\alpha x} + \tilde{A}_e \tilde{v}_1(y)\, e^{-i\alpha x} + A_e^2 v_2(y)\, e^{2i\alpha x} + \tilde{A}_e^2 \tilde{v}_2(y)\, e^{-2i\alpha x}, \tag{4.4}$$

where A_e is the equilibrium amplitude ($t \to +\infty$) and a tilde ($\sim$) denotes a complex conjugate; the expression is valid to second order in amplitude. Moreover, from formulae (11), (13), (3) and (34) of his paper, we have

$$v_1 = \operatorname{sech} y, \quad v_2 = -i \operatorname{sech}^2 y, \quad A_e = \tfrac{1}{4}(3\pi\alpha c_i)^{\frac{1}{2}} + \ldots. \tag{4.5}$$

Since c is pure imaginary, $c = ic_i$. From Drazin & Howard (1962, p. 281) we have, after omission of an incorrect factor of 2,

$$c_i = (1-\alpha^2)/\pi + \ldots. \tag{4.6}$$

Substituting (4.5) and (4.6) in (4.4) we obtain

$$A_e = \tfrac{1}{4}[3\alpha(1-\alpha^2)]^{\frac{1}{2}} + \ldots \tag{4.7}$$

and

$$v = 2A_e \operatorname{sech} y \cos \alpha x + 2A_e^2 \operatorname{sech}^2 y \sin 2\alpha x. \tag{4.8}$$

If we change phase by replacing αx in (4.8) by $(\alpha x + 3\pi/2)$ and note that α may be written in as $[1-\frac{1}{2}(1-\alpha^2)]$ approximately, we find that (4.8) and (4.7) yield (4.3) to $O(1-\alpha^2)$. To this order, we have established the equivalence of the equilibrium state towards which Schade's solution tends, with the finite-amplitude state calculated here.

There are, however, difficulties in interpretation of Schade's work. Since his expansion (4.4) is based on the linearized amplified solution at B, there is a Reynolds stress of order $c_i |A|^2$ (see, e.g. Lin 1955, p. 54 and Michalke 1964, p. 549). This persists according to (4.4) even when equilibrium is reached. But, for an inviscid flow in an equilibrium state of finite amplitude, the dissipation of fluctuation energy is zero; consequently, the Reynolds stress must convert a zero amount of energy from the mean motion to the fluctuations. In fact, as (3.20) shows, the Reynolds stress must be zero. How is the Reynolds stress $c_i |A|^2$ nullified in equilibrium? A clue is given by (2.1), (2.2) and (36) of Schade's paper: these show that the order A_e^3 terms in an expansion like (4.4) above include

$$A_e |A_e|^2 v_1^{(1)} e^{i\alpha x} + \tilde{A}_e |A_e|^2 \tilde{v}_1^{(1)} e^{-i\alpha x}, \tag{4.9}$$

together with the 3α harmonic; moreover, $v_1^{(1)}$ is complex valued. These facts indicate that there is a Reynolds-stress component of order $|A_e|^4$ in equilibrium. But, since both c_i and $|A_e|^2$ are $O(1-\alpha^2)$ in equilibrium, it can be seen that the $O(c_i |A_e|^2)$ and $O(|A_e|^4)$ contributions to the Reynolds stress are of the same order of magnitude, namely $O(1-\alpha^2)^2$. Additional terms of this order may arise also from $O(1-\alpha^2)^{\frac{3}{2}}$ term in A_e, if a_2 is included in (4.1). The upshot of this discussion is our inference that the sum total of such Reynolds-stress contributions, $O(1-\alpha^2)^2$, must be zero, because an equilibrium state is attained. Equation (3.20) of this paper gives that state, and the Reynolds stress there is zero.

Another difficulty of interpretation of Schade's work concerns the spatial distribution of vorticity. Equation (2.1) states that, in the two-dimensional inviscid flow, the total time derivative of the vorticity is zero. This means that, as a particle of fluid is convected about, its vorticity does not change. Consequently, in particular, the maximum magnitude of vorticity over the whole flow field can neither increase nor decrease with time. Let us calculate the vorticity in equilibrium, as given by Schade's work to second-order in amplitude. As implied by the equivalence established in the previous paragraph, we may calculate this from (3.20). We obtain

$$-\nabla^2\psi = -\operatorname{sech}^2 y + [3(1-\alpha^2)]^{\frac{1}{2}} \operatorname{sech}^3 y \cos\xi - \tfrac{9}{8}(1-\alpha^2)\operatorname{sech}^4 y \cos 2\xi + O(1-\alpha^2)^{\frac{3}{2}}. \tag{4.10}$$

In the basic $\tanh y$ flow, from which Schade developed the time-dependent perturbation which led to (4.10), the maximum magnitude of the vorticity occurs at $y = 0$ and is unity. On the other hand, at $\xi = \pi$, $y = 0$, (4.10) yields a value

$$|\nabla^2\psi|_{\substack{\xi=\pi \\ y=0}} = 1 + [3(1-\alpha^2)]^{\frac{1}{2}} + \tfrac{9}{8}(1-\alpha^2) + \ldots + O(1-\alpha^2)^{\frac{3}{2}}, \tag{4.11}$$

which is greater than unity since $\alpha^2 < 1$. How does this increase of vorticity magnitude come about? The answer may lie in the fact that Schade's problem is not truly inviscid, since viscosity is required at small amplitudes to eliminate the singularity (albeit in the complex y-plane, cf. Lin 1955) and to enable the non-singular development of the appropriate Reynolds stress. Presumably then, the greater maximum magnitude of the vorticity arises from the action of viscosity in smoothing out the pole in the vorticity of inviscid theory, and persists even when the explicit effect of viscosity has disappeared (in the equilibrium state). However, this suggestion does not completely settle the issue, and for further ideas on complex matters of this kind the author is referred to Michalke (1965).

As far as the perturbation scheme of this paper is concerned the difficulty does not arise, since we are asking simply about possible equilibrium states, and not how they develop in time from some other state of unstable equilibrium. It is also worth noting that the circulation around the rectangle bounded by the lines $x = 0$ and $2\pi/\alpha$, and $y = \pm\infty$, is the same for the $\tanh y$ profile as for the equilibrium flow; the circulation is, of course, an area integral of vorticity by Stokes theorem (see a discussion in part 2).

Part 2. Some exact solutions

5. Mathematical introduction

As an illustration of the studies of part 1, we here describe and comment on a class of solutions of the (non-linear) inviscid incompressible equations of motion, first of all in plane flow, but then in three dimensions.

Considering steady two-dimensional wave motions based on the hyperbolic-tangent velocity profile, we note that such flows are steady in a frame of reference which moves with the velocity of the inflexion point. In that frame the relevant differential equation (2.1) can be written, by elimination of the pressure, in the form of the vorticity equation, which states that the Jacobian of stream function (ψ) and vorticity ($-\nabla^2\psi$) is zero. Thus we know that an equation of the form

$$\nabla^2\psi = f(\psi) \tag{5.1}$$

is valid, where f is some function. Moreover the stream function for the parallel flow with the hyperbolic-tangent profile, namely $\psi = \ln \cosh y$, where y is the direction of shear and x is the direction of flow, satisfies the differential equation

$$\nabla^2\psi = e^{-2\psi}. \tag{5.2}$$

This partial differential equation is an example of Liouville's equation, in which the Laplacian of ψ is an exponential function of ψ:

$$\nabla^2\psi = A\,e^{-B\psi}. \tag{5.3}$$

The parameters A and B are supposed real.

Such equations have attracted the attentions of Liouville (1853), Poincaré (1898), Picard (1893, 1898, 1905), Lichtenstein (1913, 1915), Bieberbach (1916), Walker (1915) and Brodetsky (1924), and brief discussions are to be found in books by Bateman (1952) and Davies (1962). (The author acknowledges the help of Dr E. Varley in directing him to some of these papers.) Many exact solutions of (1.3) are known, including Liouville's general solution

$$e^{-B\psi} = \frac{8}{AB}\frac{u_x^2+u_y^2}{(u^2+v^2+1)^2} \quad (AB>0), \tag{5.4}$$

where u and v are conjugate functions and suffixes denote derivatives. Other, possibly singular, solutions include

$$e^{-B\psi} = \frac{-2}{AB}\frac{(u_x^2+u_y^2)}{u} \quad (AB<0), \tag{5.5}$$

where u is a harmonic function (Brodetsky 1924), and

$$e^{-B\psi} = \frac{-8}{AB}\frac{u_x^2+u_y^2}{(u^2+v^2-1)^2} \quad (AB<0), \tag{5.6}$$

which has not been seen by the author previously.

A solution related to (5.4) has been derived by Dr E. Varley (unpublished) in the form

$$e^{\frac{1}{2}B\psi} = \alpha_1(z)\,\overline{\alpha}_1(\overline{z})+\alpha_2(z)\,\overline{\alpha}_2(\overline{z}) \quad (AB>0), \tag{5.7}$$

where $\alpha_1(z)$, $\alpha_2(z)$ are independent analytic functions of $z(=x+iy)$ and are solutions of

$$f_{zz} = G(z)f \tag{5.8}$$

with

$$\alpha_1\alpha_2'-\alpha_2\alpha_1' = \lambda, \quad |\lambda|^2 = AB/8. \tag{5.9}$$

The function G is an arbitrary analytic function of z. An overbar denotes a complex conjugate. The correspondence between (5.4) and (5.7) may be established, though it is not completely straight-forward.

In the present paper we shall describe a class of solutions of (5.2) and, by a simple generalization, of (5.3) also. The solutions have not, to the author's knowledge, appeared in the literature previously;† they are periodic in x and

† Since the time of writing, the author has read the excellent thesis of Dr J. Schmid-Burgk (1965), who discovered these solutions (and others in the context of Plasma Physics) independently. The author is indebted to Professor A. Toomre for drawing his attention to this work.

represent inviscid shear layers which are periodic in the direction of the main flow. We shall see also that, in a certain limit, we can reproduce the mathematical solution of Laplace's equation for the flow due to a single set of point vortices, equi-spaced along a line and of equal strength, and, in another limit, the flow field of a vortex sheet. Time-dependent solutions can be obtained by translation of axes, and a class of three-dimensional flows is derivable by a simple extension.

6. The vorticity equation and an exact solution

The vorticity equation for two-dimensional motion is (2.1) where the vorticity ζ is given by

$$\zeta = -\nabla^2\psi. \tag{6.1}$$

For solutions which are steady in some frame of reference, we consider (5.1) and, motivated by the arguments of the previous section, (5.2). The solution of (5.2) that we wish to discuss is

$$\psi = \ln(C\cosh y + A\cos x), \tag{6.2}$$

where C and A (>0) are related by

$$A = (C^2-1)^{\frac{1}{2}}. \tag{6.3}$$

We shall suppose that C and A are positive. Clearly $C = 1$ gives the shear layer solution of hyperbolic-tangent form.

Although the author first encountered (6.2) by another approach, it can be derived from Liouville's solution (5.2) by use of

$$u+iv = (C-A)\tan\tfrac{1}{2}z = (C-A)\,\frac{(\sin x + i\sinh y)}{\cos x + \cosh y}, \tag{6.4}$$

where $z = x+iy$. Alternatively, we can substitute

$$G(z) = \tfrac{1}{4}, \quad \alpha_1 = (C-A)^{\frac{1}{2}}\sin\tfrac{1}{2}z, \quad \alpha_2 = (C+A)^{\frac{1}{2}}\cos\tfrac{1}{2}z \tag{6.5}$$

in Varley's solution (5.7), (5.8), to derive the same result (6.2).

There are two forms of (6.2) which warrant immediate attention.

(i) If A is small, an approximate form of (6.2), linearized in A, is

$$\psi = \ln(C\cosh y) + \epsilon\,\mathrm{sech}\,y\cos x + \ldots, \tag{6.6}$$

where

$$\epsilon = A/C. \tag{6.7}$$

The $O(\epsilon)$ perturbation could have been obtained alternatively by substitution of

$$\psi = \ln\cosh y + \phi(y)\cos\alpha x + \ldots \tag{6.8}$$

in (2.1) and linearization in ϕ, from which ϕ would satisfy the inviscid Orr–Sommerfeld equation

$$(\bar{u}-c)(\phi'' - \alpha^2\phi) - \bar{u}''\phi = 0. \tag{6.9}$$

With $\bar{u} = \tanh y$, $c = 0$, $\alpha = 1$ we have $\phi = \mathrm{sech}\,y$ (Curle 1956, Garcia 1956) as the solution tending to zero at $y = \pm\infty$, so that (6.6) is recovered. Thus the connexion of (6.2) with linearized-stability theory is established.

(ii) If C is large, (6.2) takes on the form

$$\psi - \ln C = \ln(\cosh y + \cos x), \tag{6.10}$$

which (Lamb 1932, p. 224) represents the flow due to a set of point vortices of strength, $K = -4\pi$ and spaced at a distance 2π apart on the x axis. It should be noted that (6.10) satisfies Laplace's equation, so that the vorticity is zero (except at the point vortices).

Thus, as C ranges from 1 to ∞, the flow represented by (6.2) ranges from the laminar shear layer $\bar{u} = \tanh y$ to the flow due to a set of point vortices on the axis. We shall discuss the nature of the intermediate solutions in the next section.

7. Flow patterns and vorticity distributions

The change of the flow pattern with C is most easily discussed by reference to the vorticity, which is given by

$$\zeta = -e^{-2\psi} = -[C\cosh y + A\cos x]^{-2}. \tag{7.1}$$

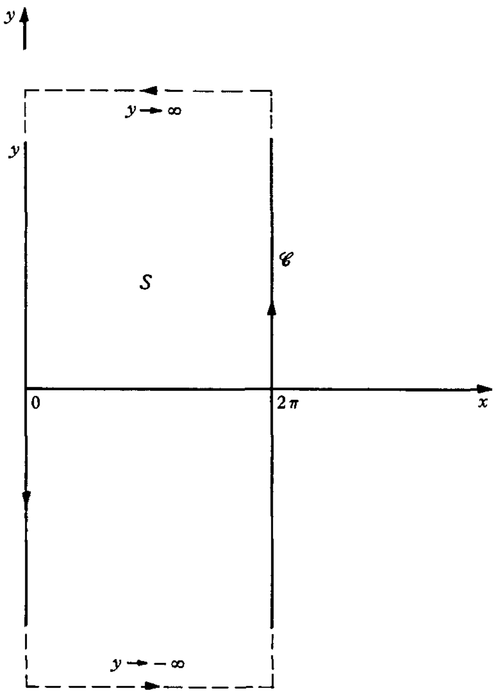

FIGURE 2. Contour for Stokes's theorem.

Consider now Stokes's theorem applied to the contour shown in figure 1. We have

$$\Gamma = \iint_S \zeta\, dx\, dy = \int_{\mathscr{C}} (u\, dx + v\, dy), \tag{7.2}$$

where S denotes the area enclosed by the contour $\mathscr{C}$. Now, from (6.2) we have

$$u = \frac{C\sinh y}{C\cosh y + A\cos x}, \quad v = \frac{A\sin x}{C\cosh y + A\cos x}. \tag{7.3}$$

Thus the contour integral depends only on the contributions at $y = \pm\infty$, since $v = 0$ on $x = 0, 2\pi$. Consequently, we have

$$\Gamma = -4\pi. \tag{7.4}$$

The result, that the circulation is constant, implies (7.2) that the area integral of the vorticity is constant. As the parameter C is varied, the total vorticity within $\mathscr{C}$ is unchanged, but the distribution of ζ with x and y changes.

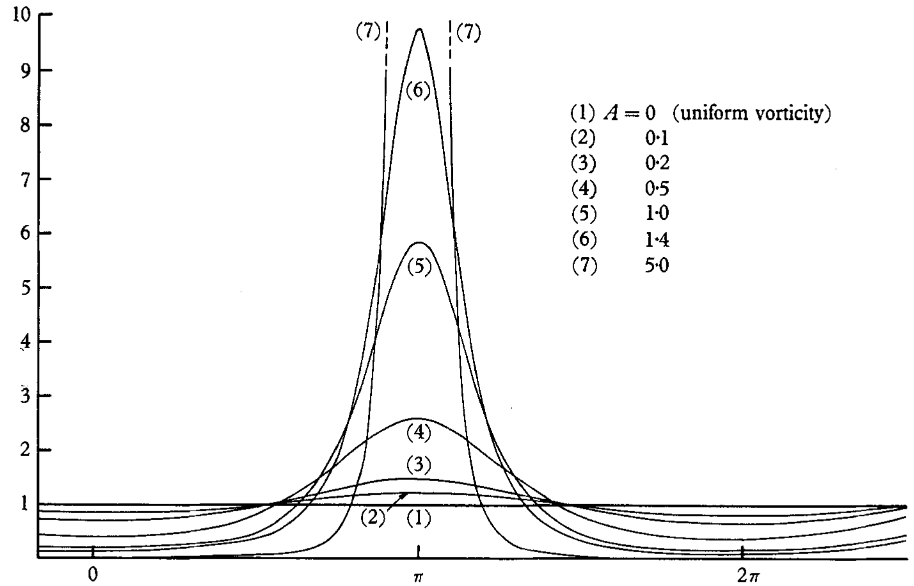

FIGURE 3. Vorticity on $y = 0$ (periodic in x).

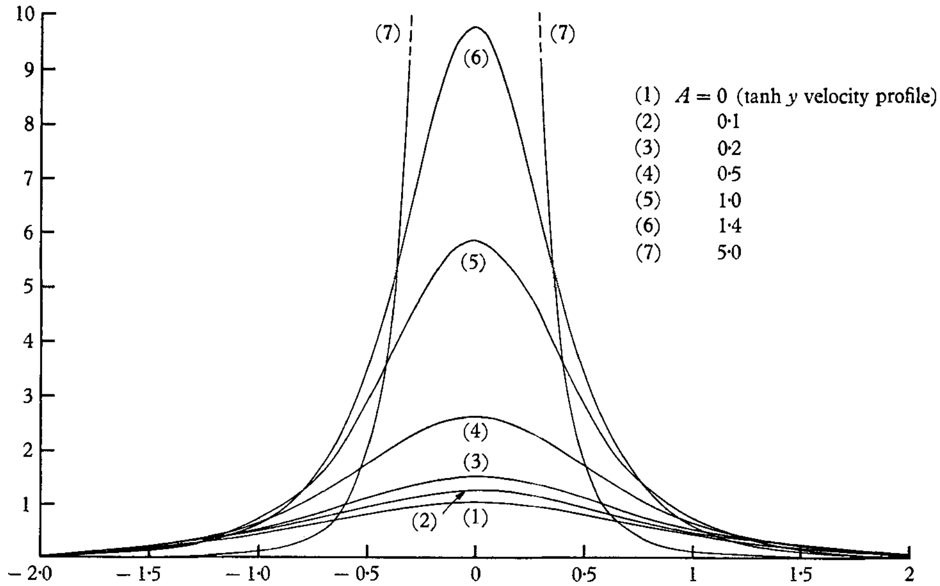

FIGURE 4. Vorticity on $x = \pi$.

In figure 3 the vorticity magnitude is shown as a function of x on $y = 0$, for different values of A. For $A = 0$, $\zeta = -1$; for small positive values of A, ζ is nearly sinusoidal in x; while for larger values of A a noticable peak of vorticity occurs at $x = (2n+1)\pi$, where n is an integer. In figure 4 we illustrate the vorticity magnitude as a function of y on $x = \pi$. As A is increased ζ develops a strong peak at $y = 0$.

This peaking of the vorticity in the neighbourhood of the points $x = (2n+1)\pi$, $y = 0$ can be illustrated mathematically as follows. From (7.1) we see that, when C and A tend to infinity, ζ tends to zero, unless x and y are chosen so that

$$C \cosh y + A \cos x \to 0. \tag{7.5}$$

Since
$$C > A > 0, \quad \cosh y \geqslant 1, \quad |\cos x| \leqslant 1, \tag{7.6}$$

the latter possibility (7.5) occurs if and only if

$$y = 0, \quad x = (2n+1)\pi. \tag{7.7}$$

At such points $\zeta = -[C-A]^{-2}$ and, in view of (6.3), this yields

$$\zeta \sim -4C^2 \quad \text{when} \quad C \to \infty. \tag{7.8}$$

Consequently, we have derived the result that, when $C \to \infty$, the vorticity is everywhere zero except at the singular points (7.7), where it tends to infinity. This establishes (6.10) and the description thereafter. We notice that in this limit, the vortex strength equals the value of Γ, as it must (7.4) since there is but one vortex within the contour $\mathscr{C}$.

The streamlines are of the celebrated 'Kelvin's cat's eyes' form, as illustrated, for example, in Lamb (1932, p. 225). The breadth of the 'cat's eyes' in the y direction depends on C, tending to zero for $C \to 1$ and to a finite, non-zero value for $C \to \infty$ (indicated by Lamb's figure.)

Other properties of the flow are also of interest and are best illustrated in another frame of reference. Suppose we write

$$x_1 = x + ct, \quad u_1 = u + c; \tag{7.9}$$

then the stream function (ψ_1) in that frame is given by

$$\psi_1 = cy + \ln[C \cosh y + A \cos(x_1 - ct)]. \tag{7.10}$$

This obviously satisfies (2.1) with x_1, ψ_1 replacing x, ψ. The corresponding velocity field is given by

$$u = c + \frac{C \sinh y}{C \cosh y + A \cos(x_1 - ct)}, \quad v = \frac{A \sin(x_1 - ct)}{C \cosh y + A \cos(x_1 - ct)}. \tag{7.11}$$

As $y \to \infty$, $u \to c+1$, whereas when $y \to -\infty$, $u \to c-1$.

For given values of x_1 and y we can consider the variation of u with time. An example, for $x_1 = 0$, $y = \frac{1}{2}$, with $C \to \infty$ is shown in figure 5. We may also define an average velocity $\bar{u}(y)$, the average being conveniently taken with respect to t at given x_1. From (7.11) we obtain

$$\bar{u} = c + \frac{C \sinh y}{[1 + C^2 \sinh^2 y]^{\frac{1}{2}}}. \tag{7.12}$$

We also define a length

$$\delta_2 = \frac{1}{4}\int_{-\infty}^{\infty} [1 - (\bar{u} - c)^2]\, dy$$
$$= \frac{1}{4A}\left[\frac{\pi}{2} + \tan^{-1}\left(\frac{A^2-1}{2A}\right)\right], \tag{7.13}$$

which, for $c = 1$ (lower fluid at rest) is equivalent to the momentum thickness defined by Browand (1966). As functions of the co-ordinate $\eta = y/\delta_2$, profiles for $C = 1, \sqrt{2}, \infty$ are illustrated in figure 6. It is seen that the curves for $C = 1$ and $\sqrt{2}$ are very close together, while that $C = \infty$ is qualitatively similar.

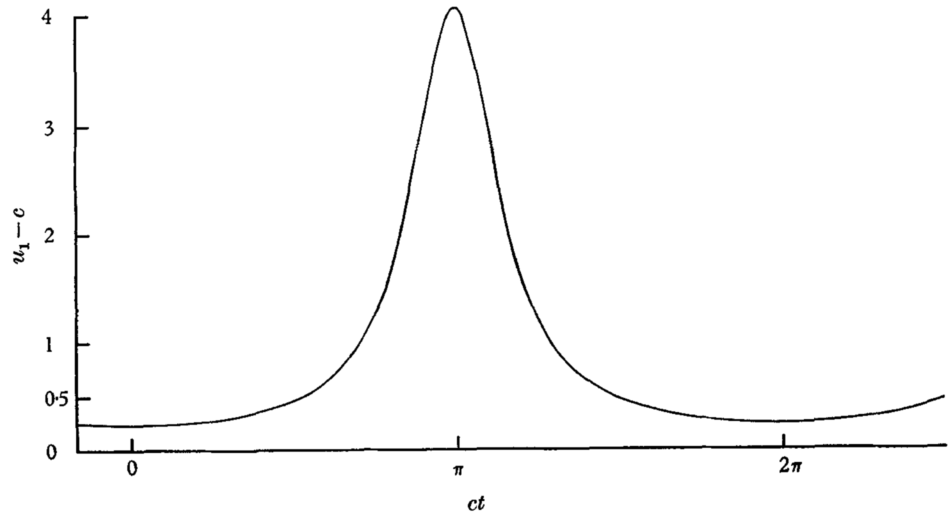

FIGURE 5. Velocity record at $y = \frac{1}{2}$.

The mean-square fluctuating velocity at a point is given by

$$\overline{u_f^2} = \frac{c}{2\pi}\int_0^{2\pi} (u-\overline{u})^2\,dt$$

$$= \frac{C^3 \sinh^2 y \cosh y}{[1+C^2\sinh^2 y]^{\frac{3}{2}}} - \frac{C^2 \sinh^2 y}{[1+C^2\sinh^2 y]}. \tag{7.14}$$

Finally we note that the solution (6.2) described in the section may be generalized to the case when the circulation around $\mathscr{C}$ is K and the wavelength is a. Then (6.2) becomes

$$\psi = \frac{-K}{4\pi}\ln\left[C\cosh\frac{2\pi y}{a} + A\cos\frac{2\pi x}{a}\right], \tag{7.15}$$

which satisfies

$$\nabla^2\psi = \frac{-\pi K}{a^2}\,e^{8\pi\psi/K}. \tag{7.16}$$

The velocity at $y = \pm\infty$ is $\mp K/2a$. The other formulae of this section are easily modified.

If, in (7.15), we let $a \to 0$, we obtain $\psi \sim K|y|/2a$, which represents two uniform streams ($u = \mp K/2a$ for $u \gtrless 0$) separated by a vortex sheet.

8. Three-dimensional flows

If we consider (inviscid) flows which are independent of z, but whose z component of velocity, w, is not zero, the stream function of the u, v velocity components satisfies (2.1) while the z component of vorticity (ζ) is given by (6.1); w satisfies the linear equation

$$\frac{\partial w}{\partial t}+\frac{\partial \psi}{\partial y}\frac{\partial w}{\partial x}-\frac{\partial \psi}{\partial x}\frac{\partial w}{\partial y}=0. \tag{8.1}$$

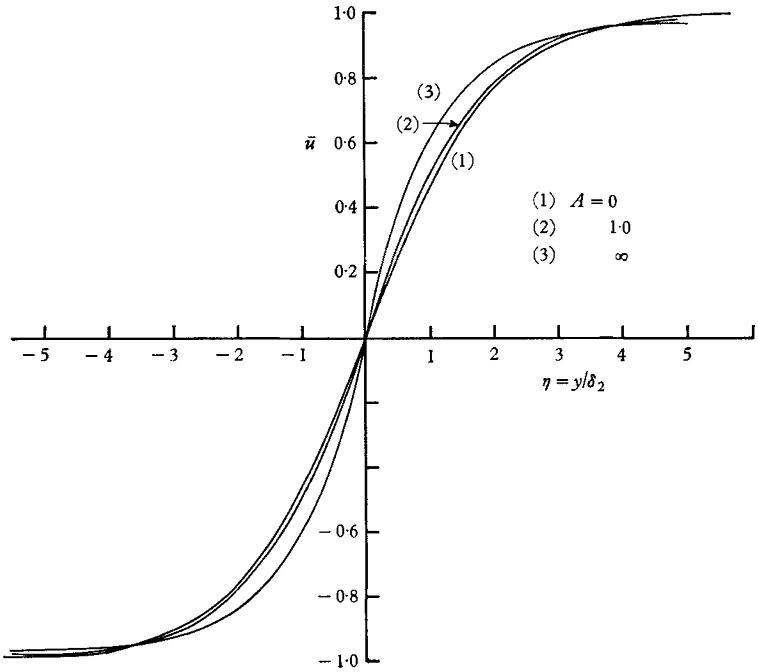

FIGURE 6. Mean velocity.

This separation of the u, v field from w is an example of the 'independence' principle, as it is known in three-dimensional boundary-layer theory (e.g. Küchemann, Crabtree & Sowerby 1963). Moreover, if ψ is independent of time, we have

$$w = g(\psi), \tag{8.2}$$

where g is an arbitrary function of ψ. Here we shall assume that ψ is given by (6.2). An important consequence of this assumption is that, since (6.2) shows ψ to be an even function of y, w is even if g is an analytic function. If on the other hand, g has a branch point this result may not be true for *all* the solutions w, since a change of branch may occur.

Three-dimensional oscillations associated with the tanh y basic velocity profile have been studied by Benney (1961) in connexion with three-dimensional developments in boundary-layer transition. We consider here cases of waves which are oblique to the direction of the basic (plane) flow, since they can be described by (2.1) and (8.1). Especially we first consider waves which are stationary (6.2), (8.2) and then later we note the transformation to travelling waves.

Such flows have the property that, when the parameter C is unity, the velocity component w is proportional to $\tanh y$. Then, since the velocity component u is also of this form, the total basic flow is proportional to $\tanh y$ and is alined in *some* direction in the (x, z)-plane, whereas perturbations from that flow are periodic in x only; this implies, as required, that the perturbations are oblique to the basic flow.

An example which may appear, at first sight, to satisfy these requirements is given by

$$g(\psi) = \gamma[1 - C^2(1 - mA/C)\, e^{-2\psi}]^{\frac{1}{2}}. \tag{8.3}$$

Using (2.3) (2.8), (4.2) we obtain from this

$$w = \gamma[1 - (1 - m\epsilon)(\cosh y + \epsilon \cos x)^{-2}]^{\frac{1}{2}}, \tag{8.4}$$

where γ and m are parameters. [The positive and negative values of (8.4) are different branches.] In order to ensure that w is real everywhere we require the expression within the square brackets to be positive. This is guaranteed by the condition

$$m > 2 - \epsilon, \tag{8.5}$$

which ensures also that w is not zero at any point.

If, in (4.4), we set $\epsilon = 0$ we obtain the continuous solution

$$w = \gamma \tanh y, \tag{8.6}$$

which involves a change of branch of (8.4) at $y = 0$. Let us now consider an expansion of (8.4) for small ϵ since, in so doing, we can make a comparison with Benney's (1961) work. We obtain first of all

$$w = \gamma \tanh y[1 + 2\epsilon \operatorname{sech} y \operatorname{cosech}^2 y \cos x + m\epsilon \operatorname{cosech}^2 y]^{\frac{1}{2}}, \tag{8.7}$$

to first order in ϵ. Further expansion is not valid at and near $y = 0$; but if we ignore this difficulty and expand for small ϵ, we obtain

$$w = \gamma \tanh y + \gamma\epsilon \operatorname{sech}^2 y \operatorname{cosech} y \cos x + \tfrac{1}{2}\gamma m\epsilon \operatorname{cosech} y \operatorname{sech} y + O(\epsilon^2). \tag{8.8}$$

In this formal expansion the lack of validity near $y = 0$ is indicated by the presence of a pole in the perturbation velocity component (w) parallel to the wave fronts. This result was first found by Benney (1961, p. 221) for inviscid three-dimensional perturbations to a plane flow. If, to put the matter another way, we write the x, y, z velocity components

$$(u, v, w) = (\tanh y, 0, \gamma \tanh y) + \epsilon(u_1, v_1, w_1)\, e^{i\alpha(x - ct)} \tag{8.9}$$

and substitute in the linearized momentum equations corresponding to (2.1) and (8.1) we obtain a neutral solution with

$$\alpha = 1, \quad c = 0, \quad w_1 = \gamma \operatorname{cosech} y \operatorname{sech}^2 y. \tag{8.10}$$

This can be shown to be equivalent to Benney's result; moreover, as far as the term periodic in x is concerned, it is equivalent to (8.8). In order to deal with this singularity in the linearized three-dimensional perturbation equations for inviscid flow, Benney naturally introduced viscosity in the neighbourhood of $y = 0$, and thereby obtained regular solutions at large Reynolds numbers. There remains, however, the possibility that the nonlinear terms, if properly included, can be used to eliminate the singularity. This situation is illustrated by a comparison of (8.4), which is finite at $y = 0$, with the expanded form (8.8), which is not. It is instructive to take this comparison further. We note first that (8.8), which is not truly a valid expanded form of (8.4), has a change from one branch

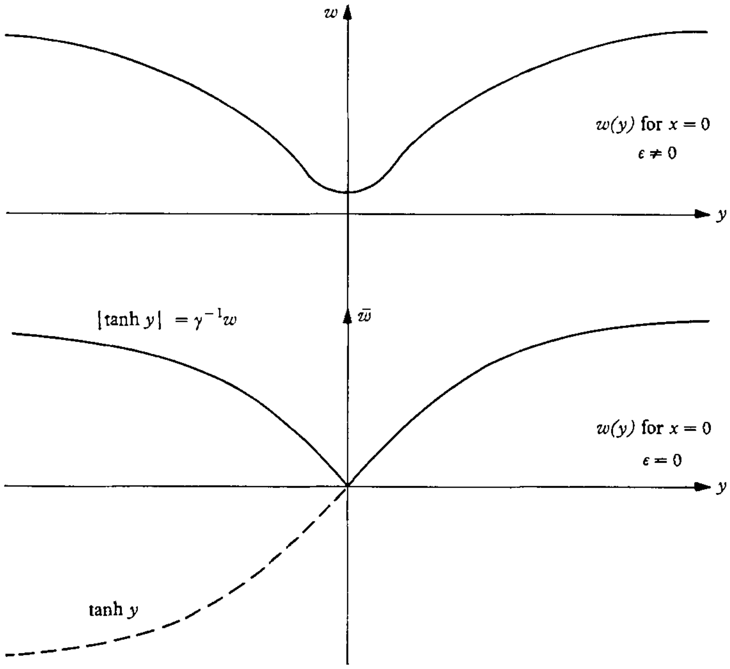

FIGURE 7. Illustration of solutions for skewed shear layer.

of (8.4) to the other at $y = 0$. On the other hand *continuous* solutions of the form (8.4) retain the same branch in general (except for the special case $\epsilon = 0$ for which (8.6), with its implied change of branch of (8.4), is the continuous solution). For $\epsilon \neq 0$ the solutions (8.4) for w are even in y, whereas that for $\epsilon = 0$ is odd. In addition the invalid perturbation (8.8) also is an odd function of y. These factors indicate that a given branch of (8.4) is not continuously connected, as the parameter ϵ is varied, with the solution (8.6) for $\epsilon = 0$. Moreover, a given branch is not related to the linearized theory perturbations of Benney's type.

At $x = 0$, (8.4) gives

$$w = \frac{\gamma(\sinh^2 y + 2\epsilon\cosh y + m\epsilon + \epsilon^2)^{\frac{1}{2}}}{\cosh y + \epsilon}. \tag{8.11}$$

As $\epsilon \to 0$ this remains even, and is illustrated schematically in figure 7, but it has a discontinuity in gradient at $y = 0$ in the limit of $\epsilon \to 0$. This contrasts with $\tanh y$, also shown in the figure.

The outcome of this discussion is that we should regard solutions (6.2) and (8.2) as representing perturbations (not necessarily small) from basic flows whose mean x components are odd, but whose mean z components are even. In the example (4.1) just discussed the latter function, $\overline{w} = \gamma\,|\tanh y|$, has a discontinuity in gradient at $y = 0$. An example without that discontinuity in gradient for $\epsilon \to 0$ is afforded by

$$g(\psi) = C^k \gamma\, e^{-k\psi}, \tag{8.12}$$

$$w = \gamma[\cosh y + \epsilon \cos x]^{-k}, \tag{8.13}$$

where k is a positive integer. The basic flow ($A = 0$) in this case is $\overline{u} = \tanh y$ in the x direction and $\overline{w} = \gamma\,\mathrm{sech}^k\, y$ in the z direction. This flow is a skewed three-dimensional shear layer.

There is an interesting difference between the properties of (8.4), (8.13). On $y = 0$ (8.4) yields

$$w(x,0) = (2\epsilon \cos x + \epsilon^2 \cos^2 x + m\epsilon)^{\frac{1}{2}}/(1+\epsilon \cos x), \tag{8.14}$$

while (8.13) yields

$$w(x,0) = \gamma(1+\epsilon \cos x)^{-k}. \tag{8.15}$$

Whereas (8.14) has scale $\epsilon^{\frac{1}{2}}$ for small ϵ, the perturbation from (8.15) has scale ϵ.

A few words about the vorticity fields of the three-dimensional flows may be helpful. From (7.1) and (8.2) the three components are

$$\xi = \frac{\partial w}{\partial y} = ug'(\psi), \quad \eta = -\frac{\partial w}{\partial x} = vg'(\psi), \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -e^{-2\psi}, \tag{8.16}$$

where a prime denotes a derivative. The balance of convection and stretching of vorticity namely that

$$\mathbf{u}.\nabla\boldsymbol{\omega} = \boldsymbol{\omega}.\nabla\mathbf{u}, \tag{8.17}$$

is such that each term is equal to

$$-\frac{1}{\rho}(\nabla p)\, g'(\psi), \tag{8.18}$$

where $\mathbf{u}$ is velocity, $\boldsymbol{\omega}$ is vorticity, p is pressure and ρ is density. Finally, we note that the flows (8.4), (8.13) remain rotational even when $\epsilon \to 1$ ($C \to \infty$) in contrast to the two-dimensional case.

Time-dependent solutions, representing waves travelling in the x direction, may be obtained by use of the transformation (7.9), and then the velocity component, say w_1, in the z direction is given by

$$w_1 = g(\psi_1 - cy), \tag{8.19}$$

ψ_1 being defined by (7.10) and g being the function introduced in (8.2). Special cases one given by (8.3) and (8.12).

9. Discussion

Experiments on instabilities in shear layers have been reported by Sato (1956), Bradshaw (1966), Browand (1966) and Freymuth (1965). In experiments the waves are periodic in time and grow in space; moreover, the shear layer thickens in the downstream direction, and the profile is not exactly of $\tanh y$ form (even in axes moving with the mean velocity at the inflexion point). In the region of instability, moreover, the mean profile is often not quite of shear-layer similarity form (Jones & Watson 1963), owing to close proximity of the plate (origin of the shear layer) and to the presence of the oscillations. Even so, observed naturally-occurring waves are found to occur approximately with the properties (of frequency, wave-number and growth rate) of the most-highly amplified disturbances of parallel-flow theory; this equivalence is found both for time-growing waves (suitably interpreted; see Browand 1966, Bradshaw 1966, Sato 1956, Michalke 1965) and in effect for spatially-growing waves, as can be seen from the fairly-close correspondence of time-growing and spatially-growing oscillations (Michalke 1965).

With regard to the profiles, as functions of y, of $(\overline{u_f^2})^{\frac{1}{2}}$, the root-mean-square fluctuation velocity (u_f) in the flow direction, Browand has commented that the 'fundamental' component of this quantity is very asymmetrical about the centre line of the shear layer in the experiments; on the other hand, he points out that neutral modes of linearized theory have a definite *anti*-symmetry when the mean profile is odd, as the experimental profiles nearly are. Consequently there is a disagreement between experiment and theory of neutral waves. However, Michalke (1965) has since found that the most-highly amplified waves (spatially) *do* have profiles of root-mean-square velocity fluctuation (fundamental component) in good agreement with Freymuth's (1965) experiments (the results of which are broadly similar to Browand's). Consequently this discrepancy between linearized theory, and the experiments in the appropriate ranges of amplitude, has been resolved.

We wish to comment, however, that the results of the present paper also show antisymmetry of the fundamental component of $(\overline{u_f^2})^{\frac{1}{2}}$ about $y = 0$, as can be seen from (3.20) when differentiated with respect to y, and from (7.14). This feature is not present in any of the measured profiles of Browand, although changes take place from the linearized-theory profiles of Michalke. Consequently it seems that equilibria of the kind discussed here were not developed in Browand's experiments.

There is a relevant comparison we can make, however; we may judge together the experimental and theoretical root-mean-square velocity fluctuations of the first harmonic when that of the fundamental is specified by the experimental evidence. In figure 10(*b*) of Browand's paper the fundamental oscillates with a maximum amplitude, $(\overline{u_f^2})^{\frac{1}{2}}_{\text{fund.}}$, of 7·6 % of the maximum flow velocity, while the first-harmonic component has maximum amplitude $(\overline{u_f^2})^{\frac{1}{2}}_{\text{harm.}}$, of 1·3 %. From (3.20) of this paper, or from (7.3), we have

$$u_f = -\epsilon \operatorname{sech} y \tanh y \cos \xi + \tfrac{1}{2}\epsilon^2 \operatorname{sech}^2 y \tanh y \cos 2\xi + O(\epsilon^3),$$

since the fluctuating part, u_f, is the same in both formulae to order (ϵ^2). Thus, to $O(\epsilon^2)$,

$$(\overline{u_f^2})^{\frac{1}{2}}_{\text{fund.}} = \frac{1}{\sqrt{2}}\epsilon \operatorname{sech} y\,|\tanh y|;\quad (\overline{u_f^2})^{\frac{1}{2}}_{\text{harm.}} = \frac{1}{2\sqrt{2}}\epsilon^2 \operatorname{sech}^2 y\,|\tanh y|.$$

The maximum value for the fundamental is $\epsilon/(2\sqrt{2})$, while that of the first harmonic is $\epsilon^2/(3\sqrt{6})$. Now we take axes moving so as to reduce the lower fluid to rest, as in the experiments and then the maximum velocity is 2. Referred to that maximum the root-mean-square components are $\epsilon/(4\sqrt{2})$ for the fundamental and $\epsilon^2/(6\sqrt{6})$ for the first harmonic. If we choose ϵ so that $\epsilon/(4\sqrt{2}) = 0{\cdot}076$ to match the experimental value (7·6 %), we obtain $\epsilon = 0{\cdot}43$; then $\epsilon^2/(6\sqrt{6}) = 0{\cdot}0126$, which is close to Browand's value (1·3 % of maximum flow velocity). This comparison is satisfactory, as is a similar one for figure 10(*c*) of Browand's paper, when we recognize that we have ignored terms of order $\epsilon^2 = 0{\cdot}185$ times each of the above values, and have ignored the lack of correspondence of experimental and theoretical wave-numbers. Consequently it seems that non-linear theories of this type do give reasonably-correct amplitude ratios between the fundamental and harmonic, for a given overall amplitude. At later stages of the experiments, the present theory does not apply since subharmonics develop (half frequency) very strongly; an explanation of this phenomenon has been advanced by Kelly (1967).

The exact solutions presented in part 2 are *possible* solutions of the inviscid equations of motion. They are not the only solutions of the periodic shear-layer type, as can be seen by comparison with 1. Moreover, they may be, and probably are, unstable in some amplitude ranges against other perturbations (Kelly 1967). For these reasons they should not be regarded as justifying any notions as to how large-amplitude (non-linear) rotational motions develop in shear layers. Especially the fact that, in certain ranges of the parameters, strong vorticity concentrations are present does not imply that such concentrations are necessarily present in experiment. But if concentrations were found to exist in experimental conditions related to the form of the solutions described here, those solutions would presumably be relevant. The author does not, however, know of any convincing experimental evidence of such concentrations.

The solutions presented here, because they are an exact consequence of certain equations, are of theoretical and illustrative value. Especially it is helpful to see a solution of Laplace's equation, namely that for the flow due to a set of point vortices equi-spaced along a line, related to a class of *rotational* flows. This result, in itself, gives added meaning to the concept of point-vortex solutions of the irrotational-flow equations. It would be of great interest to know more about the stability of this class of flows. Certain special cases have received attention: on the one hand Kelly (1967) has shown that a basic flow of the form (6.6), namely a small perturbation from the tanh y profile, is unstable against two-dimensional perturbations of double the basic x-wavelength, and his theory is in agreement with much experimental evidence; and on the other hand (Lamb 1932) the flow due to a set of point vortices also is unstable for two-dimensional disturbances of twice the basic x-wavelength, in the sense that the vortices deviate from their original positions. These two examples arise from instabilities of the extreme cases ($C \to 1$ and $C \to \infty$) of the class of solutions, and it would be of interest to know, from a stability analysis, whether the wavelength doubling phenomenon is typical for all or many members of the class.

Finally, we emphasize that many of the very important features associated with three-dimensionality in shear flows are not touched upon in this paper (cf. Benney 1961; Stuart 1965).

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